

2020 考研数学一真题及解析 (完整版)

来源: 文都教育

一、选择题: 1~8 小题, 每小题 4 分, 共 32 分. 下列每题给出的四个选项中, 只有一个选项是符合题目要求的, 请将选项前的字母填在答题纸指定位置上.

1. $x \rightarrow 0^+$ 时, 下列无穷小阶数最高的是

A. $\int_0^x (e^{t^2} - 1) dt$

B. $\int_0^x \ln(1 + \sqrt{t^3}) dt$

C. $\int_0^{\sin x} \sin t^2 dt$

D. $\int_0^{1-\cos x} \sqrt{\sin^3 t} dt$

1. 答案: D

解析: A. $\int_0^x (e^{t^2} - 1) dt \sim \int_0^x t^2 dt = \frac{x^3}{3}$

B. $\int_0^x \ln(1 + \sqrt{t^3}) dt \sim \int_0^x t^{\frac{3}{2}} dt = \frac{2}{5} x^{\frac{5}{2}}$

C. $\int_0^{\sin x} \sin t^2 dt \sim \int_0^x t^2 dt = \frac{1}{3} x^3$

D. $\int_0^{1-\cos x} \sqrt{\sin^3 t} dt \sim \int_0^{\frac{1}{2}x^2} t^{\frac{3}{2}} dt = \frac{2}{5} t^{\frac{5}{2}} \Big|_0^{\frac{1}{2}x^2} = \frac{2}{5} \cdot \left(\frac{1}{2}\right)^{\frac{5}{2}} \cdot x^5$

2. 设函数 $f(x)$ 在区间 $(-1, 1)$ 内有定义, 且 $\lim_{x \rightarrow 0} f(x) = 0$, 则 ()

A. 当 $\lim_{x \rightarrow 0} \frac{f(x)}{\sqrt{|x|}} = 0$, $f(x)$ 在 $x = 0$ 处可导.

B. 当 $\lim_{x \rightarrow 0} \frac{f(x)}{\sqrt{x^2}} = 0$, $f(x)$ 在 $x = 0$ 处可导.

C. 当 $f(x)$ 在 $x = 0$ 处可导时, $\lim_{x \rightarrow 0} \frac{f(x)}{\sqrt{|x|}} = 0$.

D. 当 $f(x)$ 在 $x = 0$ 处可导时, $\lim_{x \rightarrow 0} \frac{f(x)}{\sqrt{x^2}} = 0$.

2. 答案: B

解析: $\because \lim_{x \rightarrow 0} \frac{f(x)}{\sqrt{x^2}} = 0 \therefore \lim_{x \rightarrow 0} \frac{f(x)}{|x|} = 0 \therefore \lim_{x \rightarrow 0^+} \frac{f(x)}{x} = 0, \lim_{x \rightarrow 0^-} \frac{f(x)}{x} = 0$

$\therefore \lim_{x \rightarrow 0} \frac{f(x)}{x} = 0, \lim_{x \rightarrow 0} f(x) = 0$

$\therefore \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{f(x)}{x} = 0 = f'(0)$

$\therefore f(x)$ 在 $x = 0$ 处可导. 选 B

3. 设函数 $f(x, y)$ 在点 $(0, 0)$ 处可微, $f(0, 0) = 0, \mathbf{n} = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, -1 \right) \Big|_{(0,0)}$ 且非零向量 \mathbf{d} 与

\mathbf{n} 垂直, 则 ()

A. $\lim_{(x,y) \rightarrow (0,0)} \frac{|\mathbf{n} \cdot (x, y, f(x, y))|}{\sqrt{x^2 + y^2}} = 0$ 存在

B. $\lim_{(x,y) \rightarrow (0,0)} \frac{|\mathbf{n} \times (x, y, f(x, y))|}{\sqrt{x^2 + y^2}} = 0$ 存在

C. $\lim_{(x,y) \rightarrow (0,0)} \frac{|\mathbf{d} \cdot (x, y, f(x, y))|}{\sqrt{x^2 + y^2}} = 0$ 存在

D. $\lim_{(x,y) \rightarrow (0,0)} \frac{|\mathbf{d} \times (x, y, f(x, y))|}{\sqrt{x^2 + y^2}} = 0$

3. 答案: A

解析:

$\because f(x, y)$ 在 $(0, 0)$ 处可微. $f(0, 0) = 0$

$\therefore \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \frac{f(x, y) - f(0, 0) - f'_x(0, 0) \cdot x - f'_y(0, 0) \cdot y}{\sqrt{x^2 + y^2}} = 0$

即 $\lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \frac{f(x, y) - f'_x(0, 0) \cdot x - f'_y(0, 0) \cdot y}{\sqrt{x^2 + y^2}} = 0$

$\therefore \mathbf{n} \cdot (x, y, f(x, y)) = f'_x(0, 0)x + f'_y(0, 0)y - f(x, y)$

$\therefore \lim_{(x,y) \rightarrow (0,0)} \frac{|\mathbf{n} \cdot (x, y, f(x, y))|}{\sqrt{x^2 + y^2}} = 0$ 存在

\therefore 选 A.

4. 设 R 为幂级数 $\sum_{n=1}^{\infty} a_n r^n$ 的收敛半径, r 是实数, 则 ()

A. $\sum_{n=1}^{\infty} a_n r^n$ 发散时, $|r| \geq R$

B. $\sum_{n=1}^{\infty} a_n r^n$ 发散时, $|r| \leq R$

C. $|r| \geq R$ 时, $\sum_{n=1}^{\infty} a_n r^n$ 发散

D. $|r| \leq R$ 时, $\sum_{n=1}^{\infty} a_n r^n$ 发散

4. 答案: A

解析:

$\because R$ 为幂级数 $\sum_{n=1}^{\infty} a_n x^n$ 的收敛半径.

$\therefore \sum_{n=1}^{\infty} a_n x^n$ 在 $(-R, R)$ 内必收敛.

$\therefore \sum_{n=1}^{\infty} a_n r^n$ 发散时, $|r| \geq R$.

\therefore 选 A.

5. 若矩阵 A 经初等列变换化成 B , 则 ()

A. 存在矩阵 P , 使得 $PA=B$

B. 存在矩阵 P , 使得 $BP=A$

C. 存在矩阵 P , 使得 $PB=A$

D. 方程组 $Ax=0$ 与 $Bx=0$ 同解

5. 答案: B

解析:

$\because A$ 经初等列变换化成 B .

\therefore 存在可逆矩阵 P_1 使得 $AP_1=B$

$\therefore A=BP_1^{-1}$ 令 $P=P_1^{-1}$

$\therefore A=BP. \therefore$ 选 B.

6. 已知直线 $L_1: \frac{x-a_2}{a_1} = \frac{y-b_2}{b_1} = \frac{z-c_2}{c_1}$ 与直线 $L_2: \frac{x-a_3}{a_2} = \frac{y-b_3}{b_2} = \frac{z-c_3}{c_2}$ 相交于一点, 法

向量 $a_i = \begin{bmatrix} a_i \\ b_i \\ c_i \end{bmatrix}, i=1,2,3$. 则

A. a_1 可由 a_2, a_3 线性表示

B. a_2 可由 a_1, a_3 线性表示

C. a_3 可由 a_1, a_2 线性表示

D. a_1, a_2, a_3 线性无关

6. 答案: C

解析:

令 L_1 的方程 $\frac{x-a_2}{a_1} = \frac{y-b_2}{b_1} = \frac{z-c_2}{c_1} = t$

即有 $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} + t \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} = \alpha_2 + t\alpha_1$

由 L_2 的方程得 $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_3 \\ b_3 \\ c_3 \end{pmatrix} + t \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} = \alpha_3 + t\alpha_2$

由直线 L_1 与 L_2 相交得存在 t 使 $\alpha_2 + t\alpha_1 = \alpha_3 + t\alpha_2$

即 $\alpha_3 = t\alpha_1 + (1-t)\alpha_2$, α_3 可由 α_1, α_2 线性表示, 故应选 C.

7. 设 A, B, C 为三个随机事件, 且 $P(A) = P(B) = P(C) = \frac{1}{4}, P(AB) = 0$

$P(AC) = P(BC) = \frac{1}{12}$, 则 A, B, C 中恰有一个事件发生的概率为

A. $\frac{3}{4}$

B. $\frac{2}{3}$

C. $\frac{1}{2}$

D. $\frac{5}{12}$

7. 答案: D

解析: $P(\overline{ABC}) = P(\overline{ABUC}) = P(A) - P[A(BUC)]$

$$= P(A) - P(AB + AC)$$

$$= P(A) + P(AB) - P(AC) + P(ABC)$$

$$= \frac{1}{4} - 0 - \frac{1}{12} + 0 = \frac{1}{6}$$

$$P(\overline{BAC}) = P(\overline{BAUC}) = P(B) - P[B(AUC)]$$

$$= P(B) - P(BA) - P(BC) + P(ABC)$$

$$= \frac{1}{4} - 0 - \frac{1}{12} + 0 = \frac{1}{6}$$

$$P(\overline{CBA}) = P(\overline{CBUA}) = P(C) - P[CU(BUA)]$$

$$= P(C) - P(CB) - P(CA) + P(ABC)$$

$$= \frac{1}{4} - \frac{1}{12} - \frac{1}{12} + 0 = \frac{1}{12}$$

$$P(\overline{ABC} + \overline{ACB} + \overline{BAC}) = P(\overline{ABC}) + P(\overline{ACB}) + P(\overline{BAC})$$

$$= \frac{1}{6} + \frac{1}{6} + \frac{1}{12} = \frac{5}{12}$$

选择 D

8. 设 X_1, X_2, \dots, X_n 为来自总体 X 的简单随机样本, 其中 $P(X=0) = P(X=1) = \frac{1}{2}$, $\Phi(x)$ 表

示标准正态分布函数, 则利用中心极限定理可得 $P\left(\sum_{i=1}^{100} X_i \leq 55\right)$ 的近似值为

A. $1 - \Phi(1)$

B. $\Phi(1)$

C. $1 - \Phi(2)$

D. $\Phi(2)$

8. 答案: B

解析: 由题意 $EX = \frac{1}{2}, DX = \frac{1}{4}$

$$E\left(\sum_{i=1}^{100} X_i\right) = 100EX = 50, \quad D\left(\sum_{i=1}^{100} X_i\right) = 100DX = 25$$

由中心极限定理 $\sum_{i=1}^{100} X_i \sim N(50, 25)$

$$\therefore P\left\{\sum_{i=1}^{100} X_i \leq 55\right\} = P\left\{\frac{\sum_{i=1}^{100} X_i - 50}{5} \leq \frac{55 - 50}{5}\right\} = \Phi(1)$$

故选择 B

二、填空题：9—14 小题，每小题 2 分，共 24 分。请将解答写在答题纸指定位置上。

$$9. \lim_{x \rightarrow 0} \left[\frac{1}{e^x - 1} - \frac{1}{\ln(1+x)} \right] =$$

9. 解析：

$$\lim_{x \rightarrow 0} \left(\frac{1}{e^x - 1} - \frac{1}{\ln(1+x)} \right)$$

$$= \lim_{x \rightarrow 0} \frac{\ln(1+x) - e^x + 1}{(e^x - 1)\ln(1+x)}$$

$$= \lim_{x \rightarrow 0} \frac{\ln(1+x) - e^x + 1}{x^2}$$

$$= \lim_{x \rightarrow 0} \frac{\frac{1}{1+x} - e^x}{2x}$$

$$= -1$$

$$10. \text{ 设 } \begin{cases} x = \sqrt{t^2 + 1} \\ y = \ln(t + \sqrt{t^2 + 1}) \end{cases}, \text{ 则 } \frac{d^2 y}{dx^2} \Big|_{t=1} =$$

10. 解析：

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{1}{t + \sqrt{t^2 + 1}} \left(1 + \frac{t}{\sqrt{t^2 + 1}} \right)}{\frac{t}{\sqrt{t^2 + 1}}} = \frac{1}{t}$$

$$\frac{dy^2}{dx^2} = \frac{d\left(\frac{dy}{dx}\right)}{dx} = \frac{d\left(\frac{dy}{dt}\right)}{\frac{dx}{dt}} = \frac{-\frac{1}{t^2}}{\frac{t}{\sqrt{t^2+1}}} = -\frac{\sqrt{t^2+1}}{t^3}$$

$$\text{得 } \left. \frac{dy^2}{dx^2} \right|_{t=1} = -\sqrt{2}$$

11. 若函数 $f(x)$ 满足 $f''(x) + af'(x) + f(x) = 0 (a > 0)$, 且 $f(0) = m, f'(0) = n$, 则

$$\int_0^{+\infty} f(x) dx =$$

11. 解析:

特征方程为 $\lambda^2 + a\lambda + 1 = 0$ 特征根为 λ_1, λ_2 , 则 $\lambda_1 + \lambda_2 = -a, \lambda_1 \cdot \lambda_2 = 1$, 特征根

$$\lambda_1 < 0, \lambda_2 < 0$$

$$\begin{aligned} \int_0^{+\infty} f(x) dx &= -\int_0^{+\infty} [f''(x) + af'(x)] dx \\ &= -[f'(x) + af(x)] \Big|_0^{+\infty} \\ &= n + am \end{aligned}$$

12. 设函数 $f(x, y) = \int_0^{xy} e^{xt^2} dt$, 则 $\left. \frac{\partial^2 f}{\partial x \partial y} \right|_{(1,1)} =$

12. 解析:

$$\frac{\partial f}{\partial y} = e^{x(xy)^2} \cdot x = xe^{x^3y^2}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial \left(\frac{\partial f}{\partial y} \right)}{\partial x} = e^{x^3y^2} + 3x^3y^2e^{x^3y^2}$$

$$\left. \frac{\partial^2 f}{\partial x \partial y} \right|_{(1,1)} = e + 3e = 4e.$$

13. 行列式 $\begin{vmatrix} a & 0 & -1 & 1 \\ 0 & a & 1 & -1 \\ -1 & 1 & a & 0 \\ 1 & -1 & 0 & a \end{vmatrix} =$

13. 解析:

$$\begin{aligned} & \begin{vmatrix} a & 0 & -1 & 1 \\ 0 & a & 1 & -1 \\ -1 & 1 & a & 0 \\ 1 & -1 & 0 & a \end{vmatrix} = \begin{vmatrix} a & 0 & -1 & 1 \\ 0 & a & 1 & -1 \\ -1 & 1 & a & 0 \\ 0 & 0 & a & a \end{vmatrix} \\ & = \begin{vmatrix} 0 & a & -1+a^2 & 1 \\ 0 & a & 1 & -1 \\ -1 & 1 & a & 0 \\ 0 & 0 & a & a \end{vmatrix} = - \begin{vmatrix} a & -1+a^2 & 1 \\ a & 1 & -1 \\ 0 & a & a \end{vmatrix} \\ & = - \begin{vmatrix} a & a^2-2 & 1 \\ a & 2 & -1 \\ 0 & 0 & a \end{vmatrix} = a^4 - 4a^2. \end{aligned}$$

14. 设 X 服从区间 $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ 上的均匀分布, $Y = \sin X$, 则 $\text{Cov}(X, Y) =$

14. 解析:

$$\text{解 } f(x) = \begin{cases} \frac{1}{\pi} & -\frac{\pi}{2} < x < \frac{\pi}{2} \\ 0 & \text{其他} \end{cases}$$

$$\text{cov}(X, Y) = EXY - EXEY$$

$$= E(X \sin X) - EXE(\sin X)$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x \sin x \frac{1}{\pi} dx - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{\pi} x dx \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{\pi} \sin x dx$$

$$= 2 \frac{1}{\pi} \int_0^{\frac{\pi}{2}} x \sin x dx - 0$$

$$= \frac{2}{\pi} \int_0^{\frac{\pi}{2}} (-x) d \cos x$$

$$= \frac{2}{\pi} \left(-x \cos x \Big|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos x dx \right)$$

$$= \frac{2}{\pi} \left(0 + \sin x \Big|_0^{\frac{\pi}{2}} \right) = \frac{2}{\pi}$$

三、解答题: 15~23 小题, 共 94 分. 请将解答写在答题纸指定位置上. 解答写出文字说明、证

明过程或演算步骤.

15. (本题满分 10 分)

求函数 $f(x, y) = x^3 + 8y^3 - xy$ 的最大值

15.解析:

求一阶导可得

$$\frac{\partial f}{\partial x} = 3x^2 - y$$

$$\frac{\partial f}{\partial y} = 24y^2 - x$$

$$\text{令 } \begin{cases} \frac{\partial f}{\partial x} = 0 \\ \frac{\partial f}{\partial y} = 0 \end{cases} \text{ 可得 } \begin{cases} x = 0 \\ y = 0 \end{cases} \begin{cases} x = \frac{1}{6} \\ y = \frac{1}{12} \end{cases}$$

求二阶导可得

$$\frac{\partial^2 f}{\partial x^2} = 6x \quad \frac{\partial^2 f}{\partial x^2 y} = -1 \quad \frac{\partial^2 f}{\partial y^2} = 48y$$

当 $x = 0, y = 0$ 时, $A = 0, B = -1, C = 0$

$AC - B^2 < 0$ 故不是极值.

当 $x = \frac{1}{6}, y = \frac{1}{12}$ 时

$A = 1, B = -1, C = 4.$

$AC - B^2 > 0, A = 1 > 0$ 故 $(\frac{1}{6}, \frac{1}{12})$ 是极小值点

$$\text{极小值 } f\left(\frac{1}{6}, \frac{1}{12}\right) = \left(\frac{1}{6}\right)^3 + 8\left(\frac{1}{12}\right)^3 - 6 \times \frac{1}{12} = -\frac{1}{216}$$

16. (本题满分 10 分)

计算曲线积分 $I = \int_L \frac{4x-y}{4x^2+y^2} dx + \frac{x+y}{4x^2+y^2} dy$, 其中 L 是 $x^2 + y^2 = 2$, 方向为逆时针方向

16.解析:

$$\text{设 } P = \frac{4x-y}{4x^2+y^2}, Q = \frac{x+y}{4x^2+y^2}$$

$$\text{则 } \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} = \frac{-4x^2 + y^2 - 8xy}{(4x^2 + y^2)^2}$$

取路径 $L_\varepsilon: 4x^2 + y^2 = \varepsilon^2$, 方向为顺时针方向.

$$\begin{aligned} & \text{则 } \int_L \frac{4x-y}{4x^2+y^2} dx + \frac{x+y}{4x^2+y^2} dy \\ &= \int_{L+L_\varepsilon} \frac{4x-y}{4x^2+y^2} dx + \frac{x+y}{4x^2+y^2} dy - \int_{L_\varepsilon} \frac{4x-y}{4x^2+y^2} dx + \frac{x+y}{4x^2+y^2} dy \\ &= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy + \frac{1}{\varepsilon^2} \int_{L_\varepsilon} (4x-y) dx + (x+y) dy \\ &= \frac{1}{\varepsilon^2} \iint_D [1 - (-1)] dx dy = \frac{1}{\varepsilon^2} \cdot 2S_{D_\varepsilon} = \frac{1}{\varepsilon^2} \cdot 2 \cdot \pi \cdot \frac{\varepsilon^2}{2} = \pi. \end{aligned}$$

17. (本题满分 10 分)

设数列 $\{a_n\}$ 满足 $a_1 = 1, (n+1)a_{n+1} + 1 = \left(n + \frac{1}{2}\right)a_n$, 证明: 当 $|x| < 1$ 时幂级数 $\sum_{n=1}^{\infty} a_n x^n$ 收敛,

并求其和函数.

17. 证明: 由 $(n+1)a_{n+1} = \left(n + \frac{1}{2}\right)a_n, a_1 = 1$ 知 $a_n > 0$

$$\text{则 } \frac{a_{n+1}}{a_n} = \frac{n + \frac{1}{2}}{n+1} < 1, \text{ 即 } a_{n+1} < a_n$$

故 $\{a_n\}$ 单调递减且 $0 < a_n < 1$, 故 $|a_n x^n| < |x^n|$

当 $|x| < 1$ 时, $\sum_{n=1}^{\infty} x^n$ 绝对收敛, 故 $\sum_{n=1}^{\infty} a_n x^n$ 收敛.

$$\begin{aligned}
 S'(x) &= \left(\sum_{n=1}^{\infty} a_n x^n \right)' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n \\
 &= a_1 + \sum_{n=1}^{\infty} (n+1) a_{n+1} x^n \\
 &= 1 + \sum_{n=1}^{\infty} \left(n + \frac{1}{2} \right) a_n x^n \\
 &= 1 + \sum_{n=1}^{\infty} n a_n x^n + \frac{1}{2} \sum_{n=1}^{\infty} a_n x^n \\
 &= 1 + x \sum_{n=1}^{\infty} n a_n x^{n-1} + \frac{1}{2} S(x) \\
 &= 1 + x S'(x) + \frac{1}{2} S(x)
 \end{aligned}$$

则 $(1-x)S'(x) - \frac{1}{2}S(x) = 1$ 即 $S'(x) - \frac{1}{2(1-x)}S(x) = \frac{1}{1-x}$

解得 $S(x) = \frac{1}{\sqrt{1-x}}(-2\sqrt{1-x} + c)$

又 $S(0) = 0$ 故 $c = 2$ 因此 $S(x) = \frac{2}{\sqrt{1-x}} - 2$.

18. (本题满分 10 分)

设 Σ 为曲面 $Z = \sqrt{x^2 + y^2}$ ($\leq x^2 + y^2 \leq 4$) 的下侧, $f(x)$ 是连续函数, 计算

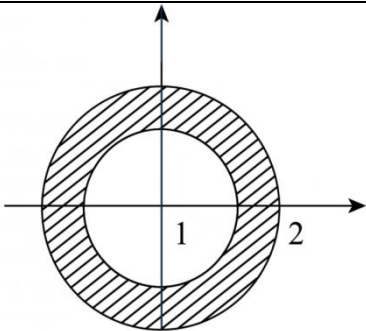
$$I = \iint_{\Sigma} [xf(xy) + 2xy - y]dydz + [yf(xy) + 2y + x]dzdx + [zf(xy) + z]dxdy$$

18. 解析:

$$z = \sqrt{x^2 + y^2} \text{ 则 } z'_x = \frac{x}{\sqrt{x^2 + y^2}}, z'_y = \frac{y}{\sqrt{x^2 + y^2}}$$

$$\text{方向余弦为 } \cos \alpha = \frac{1}{\sqrt{2}} \cdot \frac{x}{\sqrt{x^2 + y^2}}, \cos \beta = \frac{1}{\sqrt{2}} \cdot \frac{y}{\sqrt{x^2 + y^2}}, \cos \gamma = -\frac{1}{\sqrt{2}}$$

于是



$$\begin{aligned}
 I &= \frac{1}{\sqrt{2}} \iint_{\Sigma} \left\{ [xf(xy) + 2xy - y] \frac{x}{\sqrt{x^2 + y^2}} + [yf(xy) + 2y + x] \frac{y}{\sqrt{x^2 + y^2}} - [zf(xy) + z] \right\} dS \\
 &= \iint_{D_{xy}} \left(\frac{2x^2y - xy + 2y^2 + xy}{\sqrt{x^2 + y^2}} - \sqrt{x^2 + y^2} \right) dx dy \\
 &= 4 \iint_{D_1} \left(\frac{2y^2}{\sqrt{x^2 + y^2}} - \sqrt{x^2 + y^2} \right) dx dy \\
 &= 4 \left(\int_0^{\frac{\pi}{2}} d\theta \int_1^2 \frac{2r^2 \sin^2 \theta}{r} r dr - \int_0^{\frac{\pi}{2}} d\theta \int_1^2 r^2 dr \right) \\
 &= 4 \left(2 \cdot \frac{\pi}{4} \cdot \frac{7}{3} - \frac{\pi}{2} \cdot \frac{7}{3} \right) = 0.
 \end{aligned}$$

19. 设函数 $f(x)$ 在区间 $[0, 2]$ 上具有连续导数, $f(0) = f(2) = 0, M = \max_{x \in (0, 2)} \{ |f(x)| \}$,

证明 (1) 存在 $\xi \in (0, 2)$, 使得 $|f'(\xi)| \geq M$

(2) 若对任意的 $x \in (0, 2), |f'(x)| \leq M$, 则 $M = 0$.

19. 证明: (1) 由 $M = \max \{ |f(x)| \}, x \in [0, 2]$ 知存在 $c \in [0, 2]$, 使 $|f(c)| = M$,

若 $c \in [0, 1]$, 由拉格朗日中值定理得至少存在一点 $\xi \in (0, c)$, 使

$$\begin{aligned}
 f'(\xi) &= \frac{f(c) - f(0)}{c} = \frac{f(c)}{c} \\
 \text{从而 } |f'(\xi)| &= \frac{|f(c)|}{c} = \frac{M}{c} \geq M
 \end{aligned}$$

若 $c \in (1, 2]$, 同理存在 $\xi \in (c, 2)$ 使

$$\begin{aligned}
 f'(\xi) &= \frac{f(2) - f(c)}{2 - c} = \frac{-f(c)}{2 - c} \\
 \text{从而 } |f'(\xi)| &= \frac{|f(c)|}{2 - c} = \frac{M}{2 - c} \geq M
 \end{aligned}$$

综上, 存在 $\xi \in (0, 2)$, 使 $|f'(\xi)| \geq M$.

(2) 若 $M > 0$, 则 $c \neq 0, 2$.

由 $f(0) = f(2) = 0$ 及罗尔定理知, 存在 $\eta \in (0, 2)$, 使 $f'(\eta) = 0$,

当 $\eta \in (0, c]$ 时,

$$f(c) - f(0) = \int_0^c f'(x) dx$$

$$M = |f(c)| = |f(c) - f(0)| \leq \int_0^c |f'(x)| dx < Mc,$$

$$\text{又 } f(2) - f(c) = \int_c^2 f'(x) dx$$

$$M = |f(c)| = |f(2) - f(c)| \leq \int_c^2 |f'(x)| dx \leq M(2-c)$$

于是 $2M < Mc + M(2-c) = 2M$ 矛盾.

故 $M = 0$.

20. 设二次型 $f(x_1, x_2) = x_1^2 + 4x_1x_2 + 4x_2^2$ 经正交变换 $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = Q \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ 化为二次型

$g(y_1, y_2) = ay_1^2 + 4y_1y_2 + by_2^2$, 其中 $a \geq b$.

(1) 求 a, b 的值.

(2) 求正交矩阵 Q .

20. 解析:

$$(1) \text{ 设 } A = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} a & 2 \\ 2 & b \end{bmatrix}$$

由题意可知 $Q^T A Q = Q^{-1} A Q = B$.

$\therefore A$ 合同、相似于 B

$$\therefore \begin{cases} 1+4 = a+b \\ ab = 4 \end{cases} \quad a \geq b$$

$$\therefore a = 4, \quad b = 1$$

$$(2) |\lambda E - A| = \begin{vmatrix} \lambda - 1 & 2 \\ 2 & \lambda - 4 \end{vmatrix} = \lambda^2 - 5\lambda$$

$\therefore A$ 的特征值为 0, 5

当 $\lambda = 0$ 时, 解 $(0E - A)x = 0$ 得基础解为 $\alpha_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

当 $\lambda = 5$ 时, 解 $(5E - A)x = 0$ 得基础解为 $\alpha_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

又 B 的特征值也为 0, 5

当 $\lambda = 0$ 时, 解 $(0E - B)x = 0$ 得 $\beta_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \alpha_2$

当 $\lambda = 5$ 时, 解 $(5E - B)x = 0$ 得 $\beta_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \alpha_1$

对 α_1, α_2 单位化

$$\gamma_1 = \frac{\alpha_1}{|\alpha_1|} = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}, \gamma_2 = \frac{\alpha_2}{|\alpha_2|} = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{bmatrix}$$

令 $Q_1 = [\gamma_1, \gamma_2], Q_2 = [\gamma_2, \gamma_1]$

$$\text{则 } Q_1^T A Q_1 = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix} = Q_2^T B Q_2$$

故 $Q_2 Q_1^T A Q_1 Q_2^T = B$

可令

$$Q = Q_1 Q_2^T$$

$$\begin{aligned} &= \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \\ &= \begin{bmatrix} \frac{4}{5} & \frac{3}{5} \\ -\frac{3}{5} & -\frac{4}{5} \end{bmatrix} \end{aligned}$$

21. 设 A 为 2 阶矩阵, $P = (\alpha, A\alpha)$, 其中 α 是非零向量且不是 A 的特征向量.

(1) 证明 P 为可逆矩阵

(2) 若 $A^2\alpha + A\alpha - 6\alpha = 0$, 求 $P^{-1}AP$, 并判断 A 是否相似于对角矩阵.

21. 解析:

(1) $\alpha \neq 0$ 且 $A\alpha \neq \lambda\alpha$.

故 α 与 $A\alpha$ 线性无关.

则 $r(\alpha, A\alpha) = 2$

则 P 可逆.

$$AP = A(\alpha, A\alpha) = (A\alpha, A^2\alpha) = (\alpha A\alpha) \begin{pmatrix} 0 & 6 \\ 1 & -1 \end{pmatrix}$$

$$\text{故 } P^{-1}AP = \begin{pmatrix} 0 & 6 \\ 1 & -1 \end{pmatrix}.$$

(2) 由 $A^2\alpha + A\alpha - 6\alpha = 0$

$$\text{设 } (A^2 + A - 6E)\alpha = 0, (A + 3E)(A - 2E)\alpha = 0$$

由 $\alpha \neq 0$ 得 $(A^2 + A - 6E)x = 0$ 有非零解

$$\text{故 } |(A + 3E)(A - 2E)| = 0$$

$$\text{得 } |A + 3E| = 0 \text{ 或 } |A - 2E| = 0$$

若 $|A + 3E| \neq 0$ 则有 $(A - 2E)\alpha = 0$, 故 $A\alpha = 2\alpha$, 与题意矛盾

故 $|A + 3E| = 0$, 同理可得 $|A - 2E| = 0$.

于是 A 的特征值为 $\lambda_1 = -3, \lambda_2 = 2$.

A 有 2 个不同特征值, 故 A 可相似对角化

22. 设随机变量 X_1, X_2, X_3 相互独立, 其中 X_1 与 X_2 均服从标准正态分布, X_3 的概率分布为

$$P\{X_3 = 0\} = P\{X_3 = 1\} = \frac{1}{2}, Y = X_3X_1 + (1 - X_3)X_2.$$

(1) 求二维随机变量 (X_1, Y) 的分布函数, 结果用标准正态分布函数 $\Phi(x)$ 表示.

(2) 证明随机变量 Y 服从标准正态分布.

22. 解析:

$$(1) F(x, y) = P\{X_1 \leq x, Y \leq y\}$$

$$= P\{X_1 \leq x, X_3(X_1 - X_2) + X_2 \leq y, X_3 = 0\} + P\{X_1 \leq x, X_3(X_1 - X_2) + X_2 \leq y, X_3 = 1\}$$

$$= P\{X_1 \leq x, X_2 \leq y, X_3 = 0\} + P\{X_1 \leq x, X_1 \leq y, X_3 = 1\}$$

若 $x \leq y$, 则 $P\{X_1 \leq x, X_1 \leq y, X_3 = 1\} = \frac{1}{2}P\{X_1 \leq x\} = \frac{1}{2}\Phi(x)$

若 $x > y$, 则 $P\{X_1 \leq x, X_1 \leq y, X_3 = 1\} = \frac{1}{2}P\{X_1 \leq y\} = \frac{1}{2}\Phi(y)$

$$\text{故 } F(x, y) = \begin{cases} \frac{1}{2}\Phi(x)\Phi(y) + \frac{1}{2}\Phi(x), & x \leq y \\ \frac{1}{2}\Phi(x)\Phi(y) + \frac{1}{2}\Phi(y), & x > y \end{cases}$$

(2)

$$\begin{aligned} F_Y(y) &= P\{Y \leq y\} \\ &= P\{X_3(X_1 - X_2) + X_2 \leq y\} \\ &= \frac{1}{2}P\{X_3(X_1 - X_2) + X_2 \leq y \mid X_3 = 0\} + \frac{1}{2}P\{X_3(X_1 - X_2) + X_2 \leq y \mid X_3 = 1\} \\ &= \frac{1}{2}P\{X_2 \leq y \mid X_3 = 0\} + \frac{1}{2}P\{X_1 \leq y \mid X_3 = 1\} \\ &= \frac{1}{2}\Phi(y) + \frac{1}{2}\Phi(y) \\ &= \Phi(y). \end{aligned}$$

23. 设某种元件的使用寿命 T 的分布函数为

$$F(t) = \begin{cases} 1 - e^{-\left(\frac{t}{\theta}\right)^m}, & t \geq 0, \\ 0, & \text{其他.} \end{cases}$$

其中 θ, m 为参数且大于零.

(1) 求概率 $P\{T > t\}$ 与 $P\{T > s+t \mid T > s\}$, 其中 $s > 0, t > 0$.

(2) 任取 n 个这种元件做寿命试验, 测得它们的寿命分别为 t_1, t_2, \dots, t_n , 若 m 已知, 求 θ 的

最大似然估计值 $\hat{\theta}$.

23. 解析:

$$(1) P\{T > t\} = 1 - F(t) = e^{-\left(\frac{t}{\theta}\right)^m}$$

$$P\{T > s+t \mid T > s\} = P\{T > t\} = e^{-\left(\frac{t}{\theta}\right)^m}$$

$$(2) f(t) = F'(t) = \begin{cases} m\theta^{-m}t^{m-1} \cdot e^{-\left(\frac{t}{\theta}\right)^m}, & t \geq 0 \\ 0 & \text{其他} \end{cases}$$

$$\text{似然函数 } L(\theta) = \prod_{i=1}^n f(t_i, \theta) = \begin{cases} m^n \theta^{-mn} (t_1 \cdots t_n)^{m-1} e^{-\theta^{-m} \sum_{i=1}^n t_i^m} & t_i \geq 0 \\ 0 & \text{其他} \end{cases}$$

当 $t_1 \geq 0, t_2 \geq 0, \dots, t_n \geq 0$ 时

$$L(\theta) = m^n \theta^{-mn} (t_1 \cdots t_n)^{m-1} e^{-\theta^{-m} \sum_{i=1}^n t_i^m}$$

$$\text{取对数 } \ln L(\theta) = n \ln m - mn \ln \theta + (m-1) \sum_{i=1}^n \ln t_i - \theta^{-m} \sum_{i=1}^n t_i^m$$

$$\text{求导数 } \frac{d \ln(\theta)}{d\theta} = -\frac{mn}{\theta} + m \theta^{-(m+1)} \sum_{i=1}^n t_i^m$$

$$\text{令 } \frac{d \ln(\theta)}{d\theta} = 0 \text{ 解得 } \theta = \sqrt[m]{\frac{1}{n} \sum_{i=1}^n t_i^m}$$

$$\text{所以 } \theta \text{ 的最大似然估计值 } \hat{\theta} = \sqrt[m]{\frac{1}{n} \sum_{i=1}^n t_i^m}$$